GOSFORD HIGH SCHOOL

2013
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics

General Instructions

- Reading Time - 5 minutes
- Working Time - 3 hours
- Write using a blue or black pen. Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks (100)

Section I

Total marks (**10**)

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II

Total marks (**90**)

- Attempt questions 11 – 16
- Answer in the answer booklets provided, unless otherwise instructed
- Start a new booklet for each question
- All necessary working should be shown for every question
- Allow about 2 hours 45 minutes for this section

SECTION I

- **10 Marks**
- **Attempt Questions 1-10**
- **Allow about 15 minutes for this section**
- **Answer on the multiple choice answer sheet provided**

Question 1.

5.09674 correct to 3 significant figures is:

(A) 5.09 **(B)** 5.10 **(C)** 5.096 **(D)** 5.097

Question 2.

Which of the following is equivalent to $\dfrac{1}{\sqrt{7}+2\sqrt{3}}$?

(A) $7-2\sqrt{3}$

(B) $7+2\sqrt{3}$

(C) $\dfrac{\sqrt{7}-2\sqrt{3}}{-5}$

(D) $\dfrac{\sqrt{7}+2\sqrt{3}}{-5}$

Question 3.

$\dfrac{x^2-5xy}{x^2-25y^2}$ simplifies to:

(A) $\dfrac{x}{x-5y}$

(B) $\dfrac{x}{x+5y}$

(C) $\dfrac{1-x}{1-5y}$

(D) $\dfrac{x-5y}{x+25y}$

Question 4.

Two regular dice are thrown. What is the probability that the throw will **not** result in a double?

(A) $\dfrac{1}{36}$ **(B)** $\dfrac{31}{36}$ **(C)** $\dfrac{35}{36}$ **(D)** $\dfrac{5}{6}$

Question 5.

Convert 84° into radians correct to 2 decimal places.

(A) 2.93 (B) 1.46

(C) 1.47 (D) 4.81

Question 6.

The graph of the function $y = 3\sin 2x$ has:

(A) amplitude 3 and period π

(B) amplitude 2 and period $\frac{2\pi}{3}$

(C) amplitude 3 and period $\frac{\pi}{2}$

(D) amplitude $\frac{3}{2}$ and period π

Question 7.

What is the equation of the parabola with vertex (4,2) and focus (3,2)?

(A) $(x-4)^2 = 4(y-2)$ (B) $(x-4)^2 = -4(y-2)$

(C) $(y-2)^2 = 4(x-4)$ (D) $(y-2)^2 = -4(x-4)$

Question 8.

$\frac{d}{dx}(\sin 2x) =$

(A) $2\sin 2x$ (B) $2\cos 2x$

(C) $-2\cos 2x$ (D) $\frac{1}{2}\cos 2x$

Question 9.

It is known that for a particular quadratic equation, $\alpha + \beta = -\frac{5}{3}$ and $\alpha\beta = \frac{7}{3}$. The quadratic equation could be:

(A) $6x^2 + 10x + 14 = 0$ (B) $3x^2 - 5x + 7 = 0$

(C) $3x^2 + 5x - 7 = 0$ (D) $5x^2 - 7x + 3 = 0$

Question 10.

What is the angle of inclination of the line $3x + 2y = 7$ with the positive direction of the x axis?

(A) 33° 41' (B) 56° 19'

(C) 123° 41' (D) 146° 19'

Section II

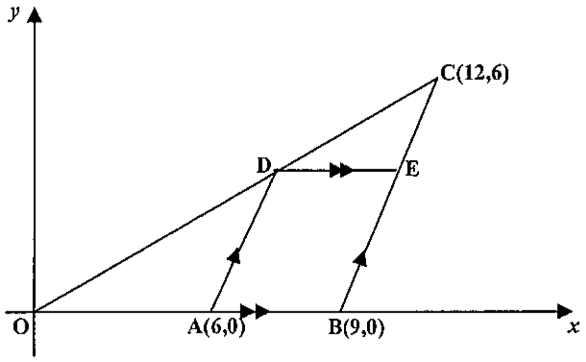
- **90 Marks**
- **Attempt Questions 11-16**
- **Allow about 2 hours 45 minutes for this section.**
- **Answer each question in a separate answer booklet.**
- **All necessary working should be shown for each question.**

Question 11.

a)	Solve $4^x = 32$	**2**
b)	Fully factorise $40 - 5y^3$	**2**
c)	Solve $\lvert 5 - 2x \rvert \geq 9$	**2**
d)	Find the exact value of $\tan\dfrac{2\pi}{3}$	**1**
e)	Differentiate $\sqrt{x}$	**1**
f)	Find the primitive function of $(x-4)^6$	**1**
g)	Find $\dfrac{d}{dx}x\cos(x+1)$	**2**
h)	State the domain and range of $y = \sqrt{3-x}$	**2**
i)	Calculate the value of $\log_5 16$, correct to 2 decimal places.	**2**

Question 12. **Start a new booklet.**

a) In the diagram, A,B and C are the points (6,0), (9,0) and (12,6) respectively. The equation of the line OC is $x-2y=0$. AD is parallel to BC and DE is parallel to the x axis.



i) Show that the equation of the line AD is $y=2x-12$. **2**

ii) Find the coordinates of the point D. **2**

iii) Prove that ΔOAD is similar to ΔDEC. **2**

b) A total of 300 tickets are sold in a raffle which has 3 prizes. There are 100 red, 100 green and 100 blue tickets. At the drawing of the raffle, winning tickets are NOT replaced before the next draw.

i) What is the probability that each of the 3 winning tickets is red? **1**

ii) What is the probability that at least one of the winning tickets is not red? **1**

iii) What is the probability that there is one winning ticket of each colour? **2**

c) Find $\int \frac{x}{x^2+5}\,dx$ **1**

d) Evaluate $\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$ **2**

e) Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ **2**

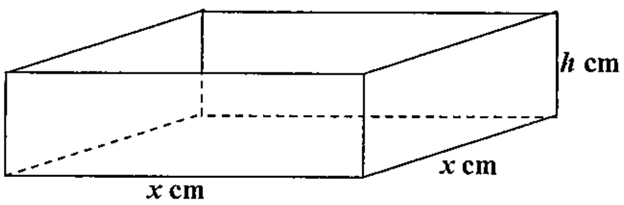
Question 13. **Start a new booklet.**

a) A circle with radius 8 cm has an arc PQ 12 cm in length. Find:

i) the size of the angle subtended by the arc PQ at the centre of the circle. **1**

ii) the length of the chord PQ, correct to 2 decimal places. **2**

b) A metal tray, in the shape of a rectangular prism with a square base, is made out of 108 square centimetres of sheet metal. The tray is open at the top. Let x cm be the side length of the base and h cm be the height as shown.



i) Show that $h = \dfrac{108 - x^2}{4x}$ **1**

ii) Show that the volume, V of the tray is given by $V = 27x - \dfrac{x^3}{4}$. **2**

iii) Find the maximum volume of the tray. **3**

c) **i)** Differentiate $\log_e(\cos x)$ with respect to x. **1**

ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \tan x dx$ **2**

d) **i)** Show that $\cos\theta \tan\theta = \sin\theta$. **1**

ii) Hence solve $8\sin\theta\cos\theta\tan\theta = \text{cosec}\theta$ for $0 \le \theta \le 2\pi$ **2**

Question 14. **Start a new booklet.**

a) Solve for x: $e^{2x} + 3e^{x} - 10 = 0$. **2**

b) Solve $2\ln x = \ln(5 + 4x)$. **2**

c) A(6, 0) and B(1, -3) are fixed points and P(x, y) moves so that PA is perpendicular to PB. Show that the locus of P is
$x^2 + y^2 - 7x + 3y + 6 = 0$ **3**

d) **i)** Sketch the graph of $y = 4\cos 2x$ for $-\pi \leq x \leq \pi$, clearly showing the x and y intercepts. **3**

ii) On the same set of axes, sketch the graph of $y = |x|$. **1**

iii) Hence state **how many** solutions there are to the equation $4\cos 2x - |x| = 0$, for $-\pi \leq x \leq \pi$. **1**

e) The curve $y = f(x)$ has gradient function $f'(x) = 3x^2 - k$, where k is a constant.

i) Find the value of k if the curve has a stationary point at (-1, 3). **1**

ii) Hence find the equation of the curve. **2**

Question 15. **Start a new booklet.**

a) Consider the function $f(x) = x^4 - 4x^3$

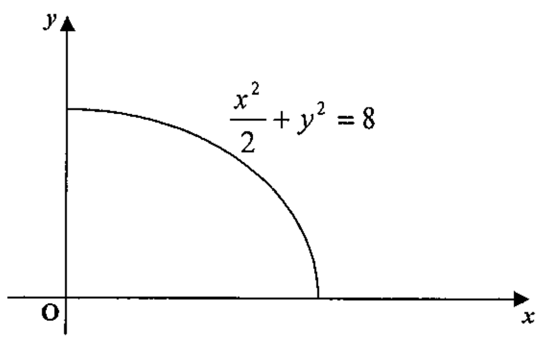
i) Show that $f'(x) = 4x^2(x-3)$ **1**

ii) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. **3**

iii) Sketch the graph of the curve $y = f(x)$, showing the stationary points **2**

iv) Find the values of x for which the graph of $y = f(x)$ is concave down. **1**

b)



The part of the curve $\frac{x^2}{2} + y^2 = 8$ that lies in the first quadrant is rotated about the x axis. Find the volume of the solid of revolution. **3**

c) Under certain climatic conditions the number N of blue-green algae satisfies the equation $N = Ae^{0.15t}$, where t is measured in days and A is a constant.

i) Show that the number of algae increases at a rate proportional to the number present. **1**

ii) When $t = 3$ the number of algae was estimated to be 1.7×10^8. Evaluate A. **2**

iii) The number of algae doubles every x days. Find x. **2**

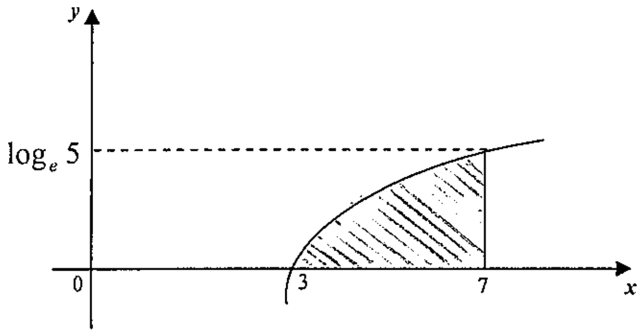
Question 16. **Start a new booklet.**

a) Find the value(s) of k for which the line $kx + y + 1 = 0$ is a tangent to the parabola $y = x^2$. **2**

b) **i)** Use Simpson's Rule with 3 function values to find an approximation to the area under the curve $y = \frac{1}{x}$ between $x = a$ and $x = 3a$, where a is positive. **2**

ii) Using the result in part i), show that $\ln 3 \approx \frac{10}{9}$. **1**

c) Part of the graph of $y = \log_e(x-2)$ is shown below.



i) Find the exact value of the area between the curve and the y axis, bounded by the lines $y = 0$ and $y = \log_e 5$. **3**

ii) Hence find the exact value of the shaded area. **1**

d) A particle is moving in a straight line. Its displacement, x metres, from the origin, O, at time t seconds, where $t \geq 0$, is given by $x = 1 - \frac{7}{t+4}$.

i) Find the initial displacement of the particle. **1**

ii) Find the velocity of the particle as it passes through the origin. **2**

iii) Show that the acceleration of the particle is always negative. **1**

iv) Sketch the graph of the displacement of the particle as a function of time. **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

MATHEMATICS TRIAL HSC 2013

SOLUTIONS

SECTION I

1) (B)

2) $\frac{1}{\sqrt{7}+2\sqrt{3}} \times \frac{\sqrt{7}-2\sqrt{3}}{\sqrt{7}-2\sqrt{3}}$

$= \frac{\sqrt{7}-2\sqrt{3}}{7-12}$ (C)

3) $\frac{x(x-5y)}{(x-5y)(x+5y)}$ (B)

4) (D)

5) $84 \times \frac{\pi}{180} = 1.47$ (C)

6) (A)

7) (D)

8) (B)

9) $x^2 + \frac{5}{3}x + \frac{7}{3} = 0$

$3x^2 + 5x + 7 = 0$ (A)

10) $2y = -3x + 7$

$y = -\frac{3}{2}x + 7$

$\tan\theta = -\frac{3}{2}$

$\theta = 123°41'$ (C)

SECTION II

11) a) $4^x = 32$

$2^{2x} = 2^5$

$2x = 5$

$x = \frac{5}{2}$

b) $40 - 5y^3 = 5(8 - y^3)$

$= 5(2-y)(4+2y+y^2)$

c) $|5-2x| \geq 9$

$5-2x \geq 9$ or $5-2x \leq -9$

$2x \leq -4$ $\quad$ $2x \geq 14$

$x \leq -2$ $\quad$ $x \geq 7$

d) $-\sqrt{3}$

e) $\frac{d}{dx} x^{1/2} = \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

f) $\frac{(x-4)^7}{7} + C$

g) $-x\sin(x+1) + \cos(x+1)$

h) Domain: $x \leq 3$

Range: $y \geq 0$

i) $\log_5 16 = \frac{\log 16}{\log 5}$

$= 1.72$

12) a) i) $M_{BC} = \frac{6}{3}$

$= 2$ $\therefore M_{AD} = 2$

$y - 0 = 2(x-6)$

$y = 2x - 12$

ii) $x - 2y = 0$ — ①

$y = 2x - 12$ — ②

sub ② into ①

$x - 2(2x-12) = 0$

$x - 4x + 24 = 0$

$3x = 24$

$x = 8$

$y = 16 - 12$

$= 4$

$\therefore$ coords of D are $(8,4)$

iii) $\angle CDE = \angle DOA$ (corresponding $\angle$'s, $DE \parallel OB$)

$\angle OAD = \angle ADE$ (alternate $\angle$'s $DE \parallel OB$)

and $\angle ADE = \angle DEC$ (alternate $\angle$'s, $AD \parallel BE$)

$\therefore \angle OAD = \angle DEC$

$\therefore \triangle OAD \parallel\!\mid \triangle DEC$ (equiangular)

b) i) $\frac{100}{300} \times \frac{99}{299} \times \frac{98}{298} = \frac{1617}{44551}$

ii) $1 - \frac{1617}{44551} = \frac{42934}{44551}$

$= 0.964$

iii) $\frac{100}{300} \times \frac{100}{299} \times \frac{100}{298} \times 6$

$= \frac{10000}{44551}$

$= 0.224$

c) $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int \frac{2x}{x^2+5}\,dx$

$= \frac{1}{2}\ln(x^2+5) + C$

d) $\int_0^{\pi/8} \sec^2 2x\,dx = \left[\frac{1}{2}\tan 2x\right]_0^{\pi/8}$

$= \frac{1}{2}\tan\frac{\pi}{4} - 0$

$= \frac{1}{2}$

e) $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

$= \int_0^4 (4-x)^{-1/2}\,dx$

$= \left[-2(4-x)^{1/2}\right]_0^4$

$= 4$

13) a) i) $L = r\theta$

$12 = 8\theta$

$\theta = \frac{3}{2}$ radians

ii) $PQ^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \times \cos 1.5$

$= 128 - 128\cos 1.5$

$= 118.9456382$

$\therefore PQ = 10.91\text{cm}$

b) i) $SA = x^2 + 4xh$

$\therefore x^2 + 4xh = 108$

$4xh = 108 - x^2$

$h = \frac{108 - x^2}{4x}$

ii) $V = x^2 h$

$= x^2 \left[\frac{108 - x^2}{4x}\right]$

$= \frac{108x - x^3}{4}$

$= 27x - \frac{x^3}{4}$

iii) $\frac{dV}{dx} = 27 - \frac{3x^2}{4}$ $\quad \frac{d^2V}{dx^2} = \frac{-6x}{4}$

for max $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

$27 - \frac{3x^2}{4} = 0$

$3x^2 = 108$

$x^2 = 36$

$x = \pm 6$

ignore -ve value as measurement

when $x = 6$, $\frac{d^2V}{dx^2} < 0$

$\therefore$ Max volume when $x = 6$.

$V = 27(6) - \frac{6^3}{4}$

$= 162 - \frac{216}{4}$

$= 108\text{cm}^3$

c) i) $\frac{d}{dx}\log(\cos x) = \frac{-\sin x}{\cos x}$

$= -\tan x$

ii) $\int_0^{\pi/4} \tan x \, dx = -\left[\ln(\cos x)\right]_0^{\pi/4}$

$= \left[\ln(\cos\frac{\pi}{4}) - \ln(\cos 0)\right]$

$= -\ln\frac{1}{\sqrt{2}} + \ln 1$

$= -\ln 2^{-1/2}$

$= \frac{1}{2}\ln 2$

d) i) $LHS = \cos\theta . \frac{\sin\theta}{\cos\theta}$

$= \sin\theta$

$= RHS$

ii) $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta$

$8\sin^2\theta = \frac{1}{\sin\theta}$

$8\sin^3\theta = 1$

$\sin^3\theta = \frac{1}{8}$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

14) a) $e^{2x} + 3e^{x} - 10 = 0$

let $u = e^x$

$u^2 + 3u - 10 = 0$

$(u+5)(u-2) = 0$

$u = -5, 2$

$\therefore e^x = -5$ or $e^x = 2$

no soln $\quad x = \ln 2$

$= 0.693$

b) $2\ln x = \ln(5+4x)$

$\ln x^2 = \ln(5+4x)$

$x^2 = 5 + 4x$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5, -1$

$\therefore x = 5$ only soln.

c) $M_{PA} \cdot M_{PB} = -1$

$M_{PA} = \frac{y}{x-6} \quad M_{PB} = \frac{y+3}{x-1}$

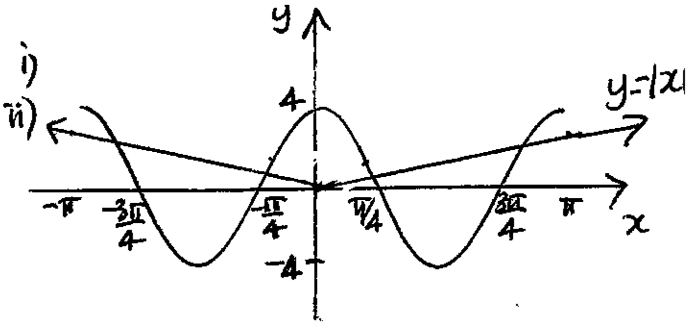
$\frac{y}{x-6} \cdot \frac{y+3}{x-1} = -1$

$y^2 + 3y = -x^2 + 7x - 6$

$x^2 + y^2 - 7x + 3y + 6 = 0$

d) i)

ii)



iii) 4

e) $f'(x) = 3x^2 - k$

i) $f'(x) = 0$, when $x = -1$

$3x^2 - k = 0$

$3 - k = 0$

$k = 3$

ii) $f'(x) = 3x^2 - 3$

$f(x) = x^3 - 3x + C$

when $x = -1, y = 3$

$3 = -1 + 3 + C$

$C = 1$

$\therefore f(x) = x^3 - 3x + 1$

15) a) $f(x) = x^4 - 4x^3$

i) $f'(x) = 4x^3 - 12x^2$

$= 4x^2(x-3)$

ii) $4x^2(x-3) = 0$

$4x^2 = 0 \quad x - 3 = 0$

$x = 0 \quad x = 3$

when $x = 0, y = 0$

$x = 3, y = -27$

$\therefore$ stat. points at $(3, -27)$ and $(0, 0)$

$f''(x) = 12x^2 - 24x$

when $x=0$, $f''(x)=0$

∴ possible point of inflexion

test for change in concavity

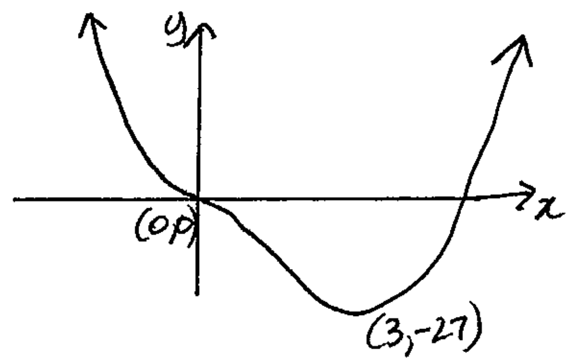
x	-1	0	1
$f''(x)$	$+$	0	$-$

∴ point of inflexion at $(0,0)$

when $x=3$, $f''(x)>0$

∴ minimum at $(3,-27)$

iii)



iv) $12x^2 - 24x < 0$

olve $12x(x-2)=0$

$x = 0, 2$

∴ concave down for

$0 < x < 2$

b) $\frac{x^2}{2} + y^2 = 8$

$y^2 = 8 - \frac{x^2}{2}$

$V = \pi \int y^2 \, dx$

$= \pi \int_0^4 8 - \frac{x^2}{2} \, dx$

$= \pi \left[8x - \frac{x^3}{6} \right]_0^4$

$= \pi \left[32 - \frac{64}{6} \right]$

$= \frac{64\pi}{3}$ cubic units

c) $N = Ae^{0.15t}$

i) $\frac{dN}{dt} = 0.15Ae^{0.15t}$

$= 0.15N$

∴ proportional to number present

ii) $1.7 \times 10^8 = Ae^{0.45}$

$A = \frac{1.7 \times 10^8}{e^{0.45}}$

$= 1.08 \times 10^8$

iii) $N = (1.08 \times 10^8) e^{0.15t}$

if doubles every x days

then $2 = e^{0.15x}$

$\ln 2 = 0.15x$

$x = 4.62$ days

16) a) $kx + y + 1 = 0$ — ①

$y = x^2$ — ②

from ① $y = -kx - 1$

∴ $x^2 = -kx - 1$

$x^2 + kx + 1 = 0$

eqn has 1 soln if line is a tangent ie $\Delta = 0$

$$k^2 - 4 = 0$$
$$k = \pm 2$$

b) i) $A \doteqdot \frac{a}{3}\left[\frac{1}{a} + \frac{1}{3a} + 4 \cdot \frac{1}{2a}\right]$

$$= \frac{a}{3}\left[\frac{3+1+6}{3a}\right]$$
$$= \frac{10}{9}$$

ii) $\int_a^{3a} \frac{1}{x}\,dx = \left[\ln x\right]_a^{3a}$

$$= \ln 3a - \ln a$$
$$= \ln 3$$

$\therefore \ln 3 \doteqdot \frac{10}{9}$

c) i) $y = \ln(x-2)$

$$x - 2 = e^y$$
$$x = e^y + 2$$

$A = \int_0^{\ln 5} e^y + 2 \; dy$

$$= \left[e^y + 2y\right]_0^{\ln 5}$$
$$= \left(e^{\ln 5} + 2\ln 5\right) - \left(e^0 + 0\right)$$
$$= 5 + 2\ln 5 - 1$$

$= 4 + 2\ln 5$ sq units

ii) $A = 7\ln 5 - (4 + 2\ln 5)$

$= 5\ln 5 - 4$ sq units

d) i) $x = 1 - \frac{7}{t+4}$

when $t = 0$, $x = 1 - \frac{7}{4}$

$$= -\frac{3}{4}$$

ie $\frac{3}{4}$ m to left of origin

ii) $x = 1 - 7(t+4)^{-1}$

$$\dot{x} = \frac{7}{(t+4)^2}$$

particle passes origin when:

$$0 = 1 - \frac{7}{t+4}$$
$$\frac{7}{t+4} = 1$$
$$7 = t + 4$$

$t = 3 \rightarrow \dot{x} = \frac{7}{(3+4)^2}$

$= \frac{1}{7}$ m/s

iii) $\ddot{x} = \frac{-14}{(t+4)^3}$

since $t \geq 0$, $(t+4)^3 > 0$

$\therefore \ddot{x}$ always $-$ve

iv)