Year 12 Trial HSC Examination - Mathematics (2U) 2006 - Solutions

Question 1

Marks

(a) Find the value of $\log_6 12$ correct to two decimal places. **2**

$$\log_6 12 = \frac{\log 12}{\log 6}$$
$$= 1.39 \text{ (to 2 d.p.)}$$

(b) The investment value of a stamp collection was originally \$24 800. If the value of the collection compounds at a rate of 5.2% annually, find its estimated value at the end of 10 years. Give your answer correct to the nearest \$100. **2**

$$\text{Value } = \$\ 24800\left(1+\frac{5.2}{100}\right)^{10}$$
$$= \$41200 \text{ (to the nearest \$100)}$$

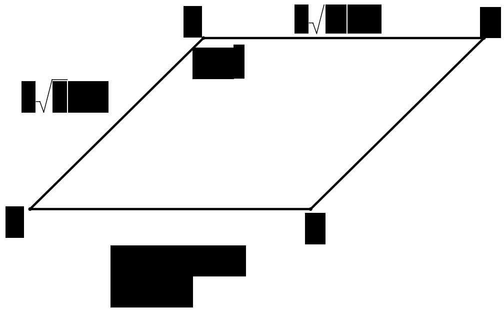
(c) The volume V of a cylinder with a base radius r and height h is given by the formula $V = \pi r^2 h$. Find the height of a cylinder when the volume is 480 cm^3 and base radius is 9.5 cm. Give your answer correct to the nearest millimetre. **2**

$$V = \pi r^2 h$$
$$h = \frac{V}{\pi r^2}$$
$$= \frac{480}{\pi \times 9.5^2}$$
$$\text{height} = 1.7 cm \text{ (to nearest mm)}$$

(d) Solve the equation $3(2x+1) - 2(3-x) = 53$. **3**

$$3(2x+1) - 2(3-x) = 53$$
$$6x + 3 - 6 + 2x = 53$$
$$8x - 3 = 53$$
$$8x = 56$$
$$x = 7$$

(e) Find the exact length of the longer diagonal in parallelogram $ABCD$. **3**



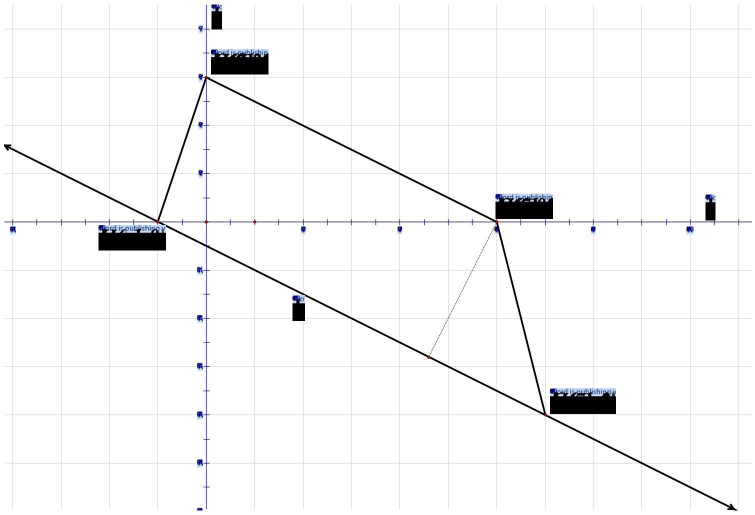
$$AC^2 = \left(2\sqrt{6}\right)^2 + \left(4\sqrt{2}\right)^2 - 2\left(2\sqrt{6}\right)\left(4\sqrt{2}\right)\cos 150°$$
$$= 24 + 32 - 16\sqrt{12} \times \left(-\frac{\sqrt{3}}{2}\right)$$
$$= 56 + 8\sqrt{36}$$
$$= 104$$
$$\text{length } AC = \sqrt{104} \text{ cm } \left(= 2\sqrt{26} \text{ cm}\right)$$

Question 2 **(Start a new page)**

Marks

The coordinates of the points E, F and G are $(0,3)$, $(6,0)$ and $(7,-4)$ respectively.

(a) Draw a neat sketch, clearly showing the above information **and** show that the line k which is parallel to EF and passes through point G, has the equation $x+2y+1=0$. **3**



$$\text{slope } EF = \frac{3-0}{0-6}$$
$$= -\frac{1}{2}$$

$$\text{Eqn. } GH: y+4 = -\frac{1}{2}(x-7)$$
$$2y+8 = -x+7$$
$$x+2y+1=0$$

(b) Find the coordinates of the point H where line k meets the x-axis. Clearly mark point H and the position line k on your diagram. **2**

when $y=0$, $x+1=0$

$\therefore x=-1$

H is $(-1,0)$

(c) Find the exact length of EF. **2**

$$EF = \sqrt{3^2+6^2}$$
$$= \sqrt{45}$$
$$\text{length } EF = 3\sqrt{5} \text{ units}$$

(d) Find perpendicular distance from point F to line GH. **2**

$$\perp \text{ dist.} = \frac{|6+0+1|}{\sqrt{1^2+2^2}} \text{ units}$$
$$= \frac{7}{\sqrt{5}} \text{ units}$$
$$= \frac{7\sqrt{5}}{5} \text{ units}$$

(e) Find the exact area of quadrilateral $EFGH$. **3**

$$GH = \sqrt{8^2 + (-4)^2}$$
$$= \sqrt{80}$$

length $GH = 4\sqrt{5}$ units

$$\text{Area} = \frac{1}{2}\left(\frac{7}{\sqrt{5}}\right)\left(3\sqrt{5} + 4\sqrt{5}\right)\text{u}^2$$
$$= 24.5\ \text{u}^2$$

Question 3 (Start a new page)

Marks

(a) Differentiate with respect to x:

(i) $\dfrac{4x^3\sqrt{x}+5}{x^2}$, **2**

$$f(x) = 4x^{1.5} + 5x^{-2}$$
$$f'(x) = 6x^{1.5} - 10x^{-3}$$
$$= 6\sqrt{x} - \frac{10}{x^3}$$

(ii) $(8-3x)^5$. **2**

$$f(x) = (8-3x)^5$$
$$f'(x) = 5(8-3x)^4 \times -3$$
$$= -15(8-3x)^4$$

(b) Find the equation of the tangent to $y = xe^{-x}$ at the point where $x = 2$. Write your answer in general form. **4**

$$y = xe^{-x}$$
$$y' = (1)(e^{-x}) + (x)(-e^{-x})$$
$$= (1-x)e^{-x}$$
When $x = 2, y = 2e^{-2}$

When $x = 2, y' = -e^{-2}$

Tangent is : $y - 2e^{-2} = -e^{-2}(x-2)$
$$y - 2e^{-2} = -e^{-2}x + 2e^{-2}$$
$$e^{-2}x + y - 4e^{-2} = 0$$
$$x + e^2 y - 4 = 0$$

(c) Differentiate $y = \dfrac{\cos x}{1+\sin x}$, hence show that $\dfrac{dy}{dx} = \dfrac{-1}{1+\sin x}$ **4**

$$\frac{dy}{dx} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$
$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$
$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$
$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$
$$= \frac{-(\sin x + 1)}{(1+\sin x)^2}$$
$$\therefore \frac{dy}{dx} = \frac{-1}{1+\sin x}$$

Question 4 (Start a new page)

Marks

(a) Evaluate

(i) $\int_0^4 \frac{dx}{3x+2} = \frac{1}{3}\left[\ln(3x+2)\right]_0^4$ **2**

$= \frac{1}{3}(\ln 14 - \ln 2)$

$= \frac{\ln 7}{3}$

(ii) $\int_{-1}^{2} \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right)x\right]_{-1}^{2}$ **2**

$= \frac{2}{\pi}\left\{\sin \pi - \sin\left(-\frac{\pi}{2}\right)\right\}$.

$= \frac{2}{\pi}\{0+1\}$

$= \frac{2}{\pi}$

(b) (i) Find the coordinates of the stationary points on the curve $y = (x+1)(x^2-8)$ and determine their nature. **5**

$y = (x+1)(x^2-8)$

$= x^3 + x^2 - 8x - 8$

$y' = 3x^2 + 2x - 8$

$= (3x-4)(x+2)$

for stat .point $y' = 0$

$\therefore\ (3x-4)(x+2) = 0$

$x = \frac{4}{3}$ or -2

if $x = \frac{4}{3},\ y = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) - 8$

$= -14\frac{14}{27}$

if $x = -2,\ y = (-2)^3 + (-2)^2 - 8(-2) - 8$

$= 4$

test nature of stat. points

$y'' = 6x + 2$

when $x = -2, \quad y'' = 6(-2) + 2$

$= -10 < 0$

$\therefore$ concave down, $\therefore$ local max. t.p. at $(-2, 4)$

when $x = \frac{4}{3}, \quad y'' = 6\left(\frac{4}{3}\right) + 2$

$= 10 > 0$

$\therefore$ concave up, $\therefore$ local min. t.p. at $\left(\frac{4}{3}, -14\frac{14}{27}\right)$

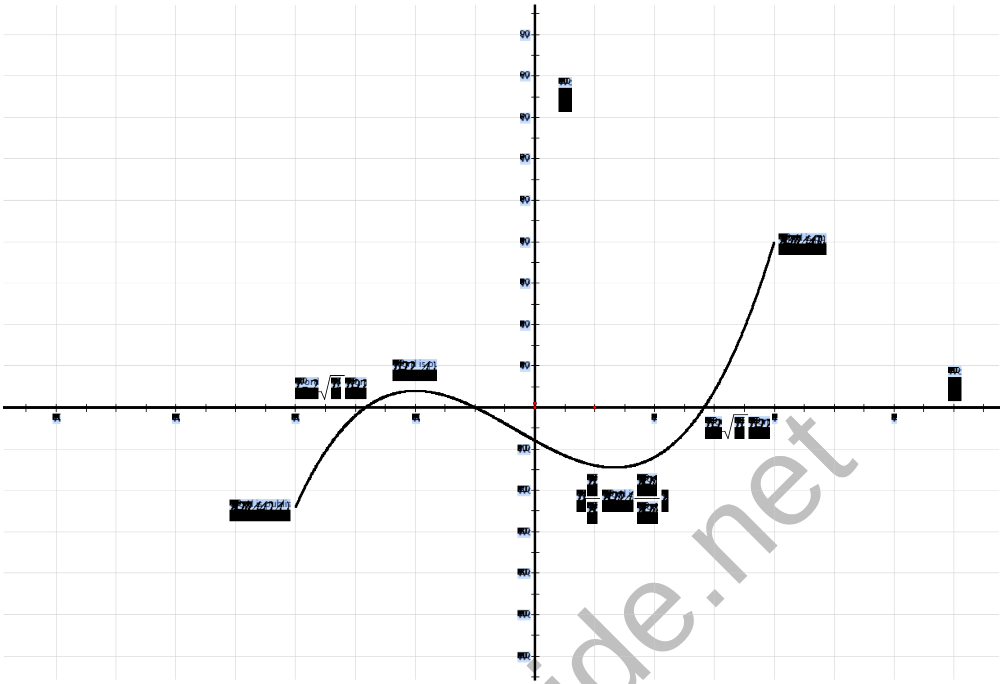
(ii) Draw a neat **half page** sketch of $y=(x+1)(x^2-8)$ in the domain $-4 \le x \le 4$. On your diagram clearly indicate the coordinates and the positions of the intercepts with the coordinate axes and the stationary points. **3**

when $x=-4, y=(-4)^3+(-4)^2-8(-4)-8$

$=-24$

when $x=4, y=(4)^3+(4)^2-8(4)-8$

$=40$

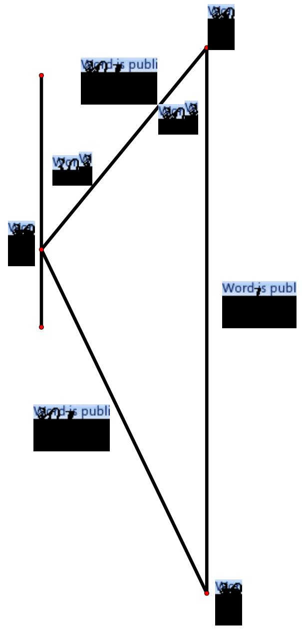


Question 5 (Start a new page)

Marks

(a) Town A is 4 km from town P and its bearing from town P is $030°T$. Town B is due south of town A and 6 km from town P.

(i) Draw a neat sketch that clearly illustrates the above information. **2**



(ii) Find the distance between towns A and B. Give your answer correct to the nearest kilometre. **4**

Let $AB = x$ km

$6^2 = 4^2 + x^2 - 2(4)(x)\cos 30°$

$36 = 16 + x^2 - 8\left(\frac{\sqrt{3}}{2}\right)x$

$36 = 16 + x^2 - 4\sqrt{3}x$

$x^2 - 4\sqrt{3}x - 20 = 0$

$x = \dfrac{4\sqrt{3} \pm \sqrt{\left(4\sqrt{3}\right)^2 - 4(1)(-20)}}{2}$

$x = \dfrac{4\sqrt{3} \pm \sqrt{48 + 80}}{2}$

$x = \dfrac{4\sqrt{3} \pm \sqrt{128}}{2}$

$x = \dfrac{4\sqrt{3} \pm 8\sqrt{2}}{2}$

$x = 2\sqrt{3} \pm 4\sqrt{2}$

distance $= \left(2\sqrt{3} + 4\sqrt{2}\right)$ km since dist. > 0

distance $= 9$ km (to nearest km)

OR

Let perpendicular distance from P to AB be y km and let the perpendicular from P meet AB at C.

$\dfrac{AC}{4} = \cos 30°$

$AC = 4\cos 30°$

$\quad = 2\sqrt{3}$

$\dfrac{PC}{4} = \sin 30°$

$PC = 4\sin 30°$

$\quad = 2$

$PC^2 + BC^2 = PB^2$

$BC^2 = 6^2 - 2^2$

$\quad = 32$

$PC = 4\sqrt{2}$

$AB = AC + BC$

$\quad = 2\sqrt{3} + 4\sqrt{2}$

$\therefore$ distance $= \left(2\sqrt{3} + 4\sqrt{2}\right)$ km

(b) (i) Find the intersection points of the line $y = 2x$ and the parabola $y = 6x - x^2$. **2**

$6x - x^2 = 2x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or 4

$x = 0, y = 0 \Rightarrow (0,0)$

$x = 4, y = 8 \Rightarrow (4,8)$

(ii) Find the area bounded by the line $y = 2x$ and the parabola $y = 6x - x^2$. **4**

$A = \int_0^4 \left[(6x - x^2) - 2x\right] dx$ or $A = \int_0^4 (6x - x^2)\, dx - \frac{1}{2} \times 4 \times 8$

$= \int_0^4 (4x - x^2)\, dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left(2 \times 4^2 - \frac{1}{3} \times 4^3\right) - 0$

$= 10\frac{2}{3}$

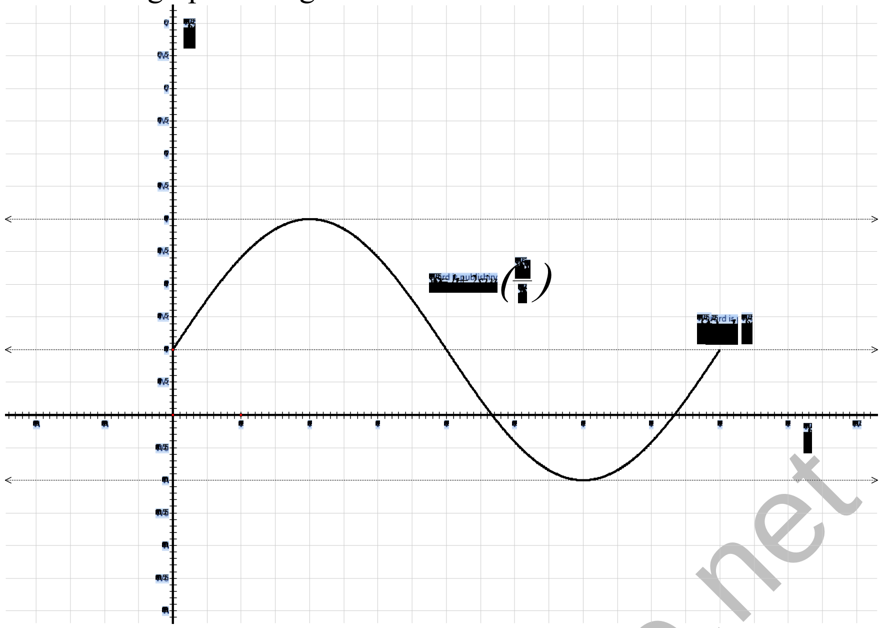
Area $= 10\frac{2}{3}\,\text{u}^2$

Question 6 (Start a new page)

Marks

(a) A particle is moving along a straight line. The particle is initially at the origin O and at time t minutes its velocity, v m/min, is given by $v = 1 + 2\sin\left(\frac{\pi}{4}t\right)$.

(i) Sketch the graph of v against t for $0 \le t \le 8$. **2**



(ii) Find the position of the particle when it first reaches its maximum speed. **4**

Max. speed when $t = 2$ (from turning point of v-t graph)

$x = t - \frac{8}{\pi}\cos\left(\frac{\pi t}{4}\right) + c$

when $t = 0, x = 0$

$\therefore\ 0 = 0 - \frac{8}{\pi}\cos(0) + c$

$c = \frac{8}{\pi}$

$\therefore\ x = t - \frac{8}{\pi}\cos\left(\frac{\pi t}{4}\right) + \frac{8}{\pi}$

when $t = 2$

$x = 2 - \frac{8}{\pi}\cos\left(\frac{2\pi}{4}\right) + \frac{8}{\pi}$

$= 2 + \frac{8}{\pi}$

disp. $= \left(2 + \frac{8}{\pi}\right)$ cm

(b) The gradient function of a curve is given by $\frac{dy}{dx} = 3 + \frac{10}{\sqrt{x}}$. **3**

Find the equation of the curve if it passes through the point $P(4,9)$.

$\frac{dy}{dx} = 3 + 10x^{-\frac{1}{2}}$

$y = 3x + 20\sqrt{x} + c$

at $P(4,9)$

$9 = 3 \times 4 + 20\sqrt{4} + c$

$c = -43$

$y = 3x + 20\sqrt{x} - 43$

(c) (i) Show that the discriminant Δ for $x^2 + mx + m$ is given by $\Delta = m^2 - 4m$. **1**

$\Delta = m^2 - 4(1)(m)$

$\quad = m^2 - 4m$

(ii) Hence, or otherwise, find the values of k for which the line $y = x + k$ and the curve $y = \frac{x}{x+1}$ do not intersect. **2**

Curves intersect when

$\frac{x}{x+1} = x + k$

$x = (x+1)(x+k)$

$x = x^2 + kx + x + k$

$x^2 + kx + k = 0$

for no points of intersection the above equation must have no real roots

i.e. $\Delta < 0$

Now $\Delta = k^2 - 4k$

$\therefore\ k^2 - 4k < 0$

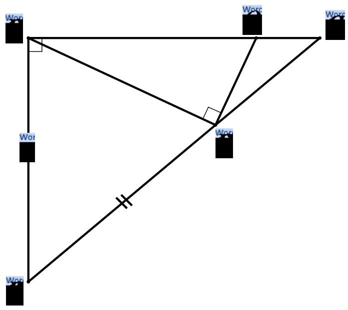
$k(k-4) < 0$

$\therefore\ 0 < k < 4$

Question 7 (Start a new page)

Marks

(a) ΔABC and ΔBPQ are right-angled triangles and $AB = AP$.



(i) Copy the diagram onto your answer sheet and prove that $\angle CBP = \angle CPQ$. (Hint: Let $\angle CBP = \alpha°$) **3**

Let $\angle CBP = \alpha°$

$\angle ABP + \alpha° = 90°$ (angle sum of right angle $ABC = 90°$)

$\angle ABP = 90° - \alpha°$

$\angle APB = 90° - \alpha°$ (equal angles are opposite equal sides in ΔABP)

$\angle CPQ + 90° + (90° - \alpha°) = 180°$ (angle sum of straight angle $APC = 180°$)

$\angle CPQ = \alpha°$

$\therefore\ \angle CBP = \angle CPQ$ (both $= \alpha°$)

(ii) Hence prove that $PC^2 = BC \times QC$ **3**

In ΔBPC and ΔPQC

$\angle CBP = \angle CPQ$ (from (i))

$\angle BCP = \angle PCQ$ (common)

$\therefore\ \Delta BPC \parallel\!| \Delta PQC$ (equiangular)

$\dfrac{PC}{QC} = \dfrac{BC}{PC}$ (ratios of corresponding sides of similar triangles are equal)

$\therefore PC^2 = BC \times QC$

* Also possible to prove that a circle passes through points *B*, *P* and *Q* and that *APC* is also a tangent to the circle, then use theorems about tangents and secants (3 unit theorem)

(b) A mass of wire cable is wound around a winch. The mass, *M* kg, of wire on the winch, at time *t* minutes, is given by the formula $M = 400 - 25\sqrt{t+100}$.

(i) Find the initial mass of wire on the winch. **2**

when $t = 0$, $M = 400 - 25\sqrt{100}$

$= 150$

Initial mass $= 150Kg$

(ii) Find the time needed to remove all the wire from the winch. **2**

when $M = 0$,

$0 = 400 - 25\sqrt{t+100}$

$25\sqrt{t+100} = 400$

$\sqrt{t+100} = 16$

$t + 100 = 256$

$t = 156$

time $= 156$ minutes $(= 2\text{hrs } 36\text{min})$

(iii) Find the initial rate at which the wire is being removed from the winch. **2**

$\frac{dM}{dt} = -25 \times \frac{1}{2}(t+100)^{-\frac{1}{2}}$

$= \frac{-25}{2\sqrt{t+100}}$

when $t = 0$

$\frac{dM}{dt} = \frac{-25}{2\sqrt{100}}$

$= -1.25$

rate $= 1.25$ kg/min

Question 8 (Start a new page)

Marks

(a) John starts work on an annual salary of $38 500. On the last day of each year for the first 10 years of employment, he will receive an annual salary increase of $1 200. For his remaining years of employment he will receive an annual increase of $1 650.

(i) What will John's annual salary be during his 6th year of employment? **2**

Years 1 to10

$\{38\,500,\ 38\,500 + 1\times 1\,200,\ 38\,500 + 2\times 1\,200,\ 38\,500 + 3\times 1\,200, ...\}$

$$A_6 = 38\,500 + 5\times 1\,200$$
$$= 44\,500$$
$$\therefore\ \text{Salary} = \$44\,500$$

(ii) What will John's annual salary be during his 15th year of employment? **2**

Years 1 to 10

$\{38\,500,\ 38\,500 + 1\times 1\,200,\ 38\,500 + 2\times 1\,200, ...,\ 38\,500 + 9\times 1\,200 (= 49300)\}$

Years 11 to 25

$\{49\,300 + 1\times 1\,650,\ 49\,300 + 2\times 1\,650,\ 49\,300 + 3\times 1\,650,\ 49\,300 + 4\times 1\,650, ...\}$

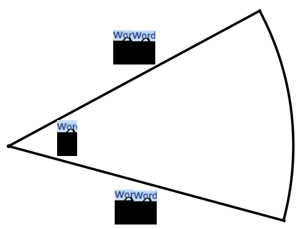
$$A_{10} = 38\,500 + 9\times 1\,200$$
$$= 49\,300$$
$$A_{15} = 49\,300 + 5\times 1650$$
$$= 57\,550$$
$$\therefore\ \text{Salary} = \$57\,550$$

(iii) How much will John's total earnings amount to at the end of his 25th year of employment? **2**

$$T = \frac{10}{2}\{2(38\,500) + 9(1\,200)\} + \frac{15}{2}\{2(49\,300) + 14(1\,650)\}$$
$$= 1\,351\,750$$

Total earnings = $1 351 750

(b) A wire framework enclosing an area of $144\ \text{m}^2$ is to be made in the shape of a sector of a circle (see diagram). The length of wire required is L metres where the circle has radius r metres and the sector angle is θ radians.



(i) Show that $L = 2r + \frac{288}{r}$. **2**

$L = 2r + r\theta$

but $\frac{1}{2}r^2\theta = 144$

$\therefore\ \theta = \frac{288}{r^2}$

$L = 2r + r \times \frac{288}{r^2}$

$L = 2r + \frac{288}{r}$

(ii) Find the minimum length of wire needed to make the framework. **4**

$L = 2r + \frac{288}{r}$

$\quad = 2r + 288r^{-1}$

$\frac{dL}{dr} = 2 - 288r^{-2}$

$\quad\quad = 2 - \frac{288}{r^2}$

for stat. point $\frac{dL}{dr} = 0$

$\frac{288}{r^2} = 2$

$2r^2 = 288$

$r^2 = 144$

$r = 12\ (\text{radius} > 0)$

test nature of stat. point

$\frac{d^2L}{dr^2} = \frac{576}{r^3}$

when $r = 12$

$\frac{d^2L}{dr^2} = \frac{576}{12^3} > 0$

$\therefore$ concave up, $\therefore$ local min. t.p.

Since the curve is continuous for $r > 0$ and there is only one turning point then the local minimum is the absolute minimum

when r = 12, $L = 2(12) + \frac{288}{12}$

$\quad\quad\quad\quad = 48$

$\therefore$ Minimum length = 48m

Question 9 (Start a new page)

			Marks
(a)	Point T is 6 cm from the centre O of a circle with radius 3 cm. Tangents drawn from point T touch the circle at points M and N and the tangents are perpendicular to the radii at these points of contact.		
	(i)	Find the exact size of $\angle MOT$.	2
	(ii)	Find the exact area of the minor sector MON.	3
(b)	(i)	Sketch the curve $y = 2 + \sqrt{x}$ and clearly shade the area bounded by the curve, the coordinate axes and the line $x = 9$.	2
	(ii)	Find the exact volume of the solid formed when the above area is rotated one revolution about the ***y*-axis**.	5

Question 10 (Start a new page)

			Marks
(a)	In order to enter a university each student is required to sit for a theory examination. If a student passes the theory examination they are then required to sit for a practical examination. It is estimated that the probability for a student to pass the theory examination is 0.8 and to pass the practical examination is 0.75.		
	(i)	Find the probability that a student chosen at random will gain entry to the university.	2
	(ii)	Find the probability that if two students chosen at random only one will gain entry to the university.	2
(b)	At 6am a population was observed to be undergoing exponential growth and after time t hours the number of individuals N_G was given by $N_G = 100e^{0.3t}$. When the population reached 500, a virus attacked the population and from that time the population underwent exponential decay so that t hours after the entry of the virus, the number of individuals N_D present was given by $N_D = 500e^{kt}$ for some constant $k < 0$. When the population fell to its original value it was decreasing at a rate of 15 individuals per hour.		
	(i)	At what time of the day did the population reach 500? Give your answer correct to the nearest minute.	3
	(ii)	Find the value of the constant k.	2
	(iii)	At what time of the day did the population return to its original value?	3

This is the END of the examination paper